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On vacuum-vacuum amplitude and Bogoliubov coefficients

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Abstract

Even if the electromagnetic field does not create pairs, virtual pairs lead to the appearance of a phase in vacuum-vacuum amplitude. This makes it necessary to distinguish the in- and out-solutions even when it is commonly assumed that there is only one complete set of solutions as, for example, in the case of a constant magnetic field. Then in- and out-solutions differ only by a phase factor which is in essence the Bogoliubov coefficient. The propagator in terms of in- and out-states takes the same form as the one for pair creating fields. The transition amplitude for an electron to go from an initial in-state to out-state is equal to unity (in diagonal representation). This is in agreement with Pauli principal: if in the field there is an electron with given (conserved) set of quantum numbers, virtual pair cannot appear in this state. So even the phase of transition amplitude remains unaffected by the field. We show how one may redefine the phases of Bogoliubov coefficients in order to express the vacuum-vacuum amplitude through them.

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1 Introduction

We use the solutions with conserved quantum numbers and do not consider radiation processes. Then the events in a cell with quantum numbers n are independent of events in cells with different quantum numbers. In other words we work in the diagonal representation. The knowledge of Bogoliubov coefficients is sufficient for obtaining the probability of any process in an external field (disregarding the radiation processes) [1-3]. But the real part of action integral W, defining the vacuum-vacuum amplitude

$$< 0_{out}|0_{in} >= e^{iW}, \quad W = \int d^4x\mathcal{L}, \tag{1}$$

is not directly expressed through Bogoliubov coefficients. At the same time some effects connected with ReW are observable. So the Lagrange function $\mathcal{L}$ of a slowly varying field defines the dielectric permittivity and magnetic permeability of the field [4-5].

The Lagrange function of the constant electromagnetic field in one loop approximation was obtained in [6-8] and in two loop approximation in [9]. Studying a model of particle production, B.DeWitt noted that ReW can be expressed through Bogoliubov coefficients with the natural choice of their phases. What we want is to choose these phases at any cost in such a way that ReW can be expressed through these phases. We show that for the constant electric field and particles with spins 0 and 1/2 the natural choice would be sufficient if it were not for the necessity to make renormalizations. For vector boson with gyromagnetic ratio $g = 2$ the situation is more complicated even for the case of a constant electric field.

We note here that transition *amplitude* for an electron to go from an in-state to out-state turns out to be unity. To show this we write down the Bogoliubov transformations and the relation between $< 0_{n\ out}|$ and $< 0_{n\ in}|$ [2]

$$a_{n\ out} = c_{1n}a_{n\ in} - c^*_{2n}b^+_{n\ in},$$
$$b^+_{n\ out} = c_{2n}a_{n\ in} + c^*_{1n}b^+_{n\ in},$$
$$< 0_{n\ out}| =< 0_{n\ in}|(c^*_{1n} - c_{2n}a_{n\ in}b_{n\ in}), \quad |c_{1n}|^2 + |c_{2n}|^2 = 1. \tag{1a}$$

From the third relation in (1a) we have eq. (28) below and from the first one

$$a^+_{n\ in} = c^{*-1}_{1n}[a^+_{n\ out} + c_{2n}b_{n\ in}].$$

Using this relation and anticommutator $\{a_{n'\ out}, a^+_{n\ out}\} = \delta_{n',n}$, we find [2]

$$< 0_{n\ out}|a_{n\ out}a^+_{n\ in}|0_{n\ in} >= c^{*-1}_{1n} < 0_{n\ out}|0_{n\ in} >= 1. \tag{1b}$$

The Pauli principle prohibits virtual pair creation in the state occupied by the electron. So even the phase of scattering amplitude remains unchanged. In particular, for the constant magnetic field $c_{2n} = 0$ but we cannot assume $c_{1n} = 1$ without violating eq. (1b) and eqs. (28-29) below because $W \neq 0$ [4-5]. In other words, even when $c_{2n} = 0$ the in- and out-vacuum are different. (This is in contrast with the remark after eq. (15) in [10].) So the Bogoliubov coefficient c_{1n} have to be coordinated with vacuum-vacuum amplitude. For the constant electromagnetic field we represent the action integral $W = \int d^4\mathcal{L}(x) = \Sigma_n W_n$ as a sum over the set of quantum numbers n. Then W_n define (in general complex) phase of Bogoliubov coefficient.

2 Scalar particle in the constant electromagnetic field

For the case of set of wave functions with conserved quantum numbers n the Bogoliubov transformation have the form

$$
\begin{aligned}
{}_{+}\psi_n &= c_{1n}\,{}^{+}\psi_n + c_{2n}\,{}^{-}\psi_n, \\
{}_{-}\psi_n &= c^*_{2n}\,{}^{+}\psi_n + c^*_{1n}\,{}^{-}\psi_n; \\
|c_{1n}|^2 &- |c_{2n}|^2 = 1.
\end{aligned}
\tag{2}
$$

Here ${}_{\pm}\psi_n = {}_{\pm}\psi_{n\;in}, {}^{\pm}\psi_n = {}^{\pm}\psi_{n\;out}$. $\pm$ denote the frequency sign. We are free to choose the phase of c_{1n} by redefining ψ_n. Indeed, if we substitute

$$
{}_{\pm}\psi_n = e^{\pm if}\,{}_{\pm}\psi_n^{new}, \quad {}^{\pm}\psi_n = e^{\mp if}\,{}^{\pm}\psi_n^{new}, \quad c_{1n} = e^{i2f} c_{1n}^{new},
$$

then eq. (2) and the propagator [2, 11]

$$
G_0(x,x') = i\Sigma_n c^{*-1}_{1n} \begin{cases} {}^{+}\psi_n(x)\,{}_{+}\psi^*_n(x'), & t > t', \\ {}_{-}\psi_n(x)\,{}^{-}\psi^*_n(x'), & t < t' \end{cases} \tag{2a}
$$

retain its form.

For definiteness we assume that the particle charge is $e' = -e, e = |e|$. Then for the constant electric field we have [2] $(n = (p_1, p_2, p_3),\ A_\mu = -\delta_{\mu 3} E t)$

$$
c_{1n} = \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} - i\varkappa)} \exp(-\frac{\pi\varkappa}{2} + i\frac{\pi}{4}),\ c_{2n} = \exp(-\pi\varkappa - i\frac{\pi}{2}), \quad \varkappa = \frac{m^2 + p_1^2 + p_2^2}{2eE}. \tag{3}
$$

We note now that in a weak electric field $|c_{2n}|$ is exponentially small and may be neglected. Then in- and out-states differ only by a phase factor. The same should be true for the magnetic field where $c_{2n} = 0$ exactly and $\ln c^*_{1n}$ should be determined.

The probability amplitude that the vacuum in the state n remains vacuum is [2]

$$
< 0_{n\;out} | 0_{n\;in} > = {c^*_{1n}}^{-1}. \tag{4}
$$

The total vacuum-vacuum amplitude is

$$
< 0_{out} | 0_{in} > = \prod_n {c^*_{1n}}^{-1} = e^{iW_0},\ W_0 = \sum_n W_{0n},\ W_{0n} = i \ln c^*_{1n} \tag{5}
$$

As we shall see below, c^*_{1n} in (4) and (5) should be replaced by C^{*ren}_{1n}. This is the renormalization of c^*_{1n}. From (3) we have

$$
\ln c^*_{1n} = \frac{1}{2}\ln 2\pi - \frac{\pi\varkappa}{2} - \frac{i\pi}{4} - \ln\Gamma(\frac{1}{2} + i\varkappa). \tag{6}
$$

As shown in [2], the vacuum-vacuum probability $| < 0_{out} | 0_{in} > |^2$ obtained from (5) and (3) agrees with Schwinger result [8]. This means that $\mathrm{Im} W_0$ is given correctly by (5) and (3). To find $\mathrm{Re} W_0$, we first consider the asymptotic representation, see eq. (1.3.12) in [12]

$$
\ln\Gamma(\frac{1}{2} + i\varkappa) = i\varkappa[\ln(i\varkappa) - 1] + \frac{1}{2}\ln 2\pi + \sum_{k=1} \frac{B_{2k}(\frac{1}{2})}{2k(2k-1)} (i\varkappa)^{1-2k}. \tag{7}
$$

(Letting k run to ∞, we may say that the r.h.s. of (7) in a certain sense exactly represent the l.h.s.; the information encoded in the r.h.s. can be decoded [13].) From (6) and (7) it follows

$$\ln c_{1n}^* = -i\{\varkappa(\ln\varkappa - 1) + \frac{\pi}{4} + \sum_{k=1} \frac{(-1)^k B_{2k}(\frac{1}{2})}{2k(2k-1)\varkappa^{2k-1}}\}. \tag{8}$$

This asymptotic expansion contains only the imaginary part of $\ln c_{1n}^*$ or only real part of W_0. It is seen from (8) that as a first step we have to go from $\ln c_{1n}^*$ to

$$\ln C_{1n}^* = \ln c_{1n}^* + i\{\varkappa(\ln\varkappa - 1) + \frac{\pi}{4}\} \tag{9}$$

in order to have $\ln C_{1n}^* \to 0$ for $\varkappa \to \infty$ (i.e. for $E \to 0$). Due to the necessity of charge renormalization we have to make the second step and introduce

$$\ln C_{1n}^{*ren} = \ln c_{1n}^* + i\{\varkappa(\ln\varkappa - 1) + \frac{\pi}{4} + \frac{1}{24\varkappa}\}. \tag{10}$$

In other words we include in $\ln C_{1n}^{*ren}$ also the term with $k = 1$ in (8). Then we have the following asymptotic representation

$$\ln C_{1n}^{*ren} = -i\sum_{k=2} \frac{(-1)^k B_{2k}(\frac{1}{2})}{2k(2k-1)\varkappa^{2k-1}}. \tag{11}$$

Summing (11) over n according to the rule

$$\sum_k \to \int \frac{d^3pL^3}{(2\pi)^3}, \quad \int dp_3 \to eET, \tag{12}$$

and making renormalization [8], we obtain the correct asymptotic representation for $\mathrm{Re}\mathcal{L}_0$

$$\mathrm{Re}\mathcal{L}_0 = \frac{1}{2}E^2 + \frac{(eE)^2}{16\pi^2}\sum_{k=2} \frac{(-1)^k B_{2k}(\frac{1}{2})}{k(k-1)(2k-1)\varkappa_0^{2k-2}}, \quad \varkappa_0 = \frac{m^2}{2eE}. \tag{13}$$

To simplify formulas and minimize confusion with T in eq.(50) we often put $L = T = 1$ in the expressions like (12). Besides, in the following we drop the Maxwell part of the Lagrangian ($\frac{1}{2}E^2$ in this case).

Now we show that the expression (9) can be brought to the form suggested by the proper-time formalism:

$$\ln C_{1n}^* \equiv \ln\sqrt{2\pi} + \eta(\ln\eta - 1) - \ln\Gamma(\frac{1}{2} + \eta) = -F(\eta),$$

$$F(\eta) = \frac{1}{2}\int\limits_0^\infty \frac{d\theta}{\theta} e^{-2\eta\theta}[\frac{1}{\sinh\theta} - \frac{1}{\theta}]. \quad \eta = i\varkappa. \tag{14}$$

Differentiating (14) with respect to η and using eq. (2.4.225) in [16], we see that the results on the left- and the right-hand sides coincide. Besides, both sides have the same asymptotic behavior for $\eta \to \infty$. So we have

$$\ln C_{1n}^* = -\frac{1}{2}\int_0^\infty \frac{ds}{s\sinh\theta} e^{-is(m^2+p_\perp^2)}[1 - \frac{\sinh\theta}{\theta}], \theta = eEs, p_\perp = p_1^2 + p_2^2. \tag{15}$$

Next, we note that the term $\frac{i}{24\kappa}$ in (10) can be written as

$$\frac{i}{24\varkappa} = -\frac{1}{12}\int_0^\infty d\theta e^{-i2\varkappa\theta}. \tag{16}$$

Hence

$$\ln C_{1n}^{*ren} = -\frac{1}{2}\int_0^\infty \frac{ds}{s\sinh\theta}e^{-is(m^2+p_\perp^2)}R(\theta), \quad R(\theta) = 1 - \sinh\theta\left(\frac{1}{\theta} - \frac{\theta}{6}\right). \tag{17}$$

Here $R(\theta)$ is a "regularizer." It is independent of quantum numbers n and is the same as in the proper-time representation of the Lagrange function [8].

Now we consider the case when there is a constant magnetic field collinear with the constant electric field. Then

$$\ln C_{1n}^{*ren}(E,H) = -\frac{1}{2}\int_0^\infty \frac{ds}{s\sinh\theta}e^{-is(m^2+eH(2l+1))}R(\theta,\tau), \tau = eHs, l = 0, 1, \cdots \tag{18}$$

and we assume that $R(\theta,\tau)$ may be obtained by the same reasoning as in [8] (or simply taken from [8])

$$R(\theta,\tau) = 1 - \sinh\theta\sin\tau\left(\frac{1}{\theta\tau} + \frac{1}{6}\frac{H^2-E^2}{EH}\right), \quad \tau = eHs, \quad \theta = eEs. \tag{19}$$

Integrating over p_3, we get (see (12) with $T=1$)

$$\int dp_3 \ln C_{1n}^{*ren}(E,H) = -\frac{1}{2}eE\int_0^\infty \frac{ds}{s\sinh\theta}e^{-is(m^2+eH(2l+1))}R(\theta,\tau). \tag{20}$$

In this expression we can turn off the electric field

$$\int dp_3 \ln C_{1n}^{*ren}(E=0,H) = -\frac{1}{2}\int_0^\infty \frac{ds}{s^2}e^{-is(m^2+eH(2l+1))}R(0,\tau),$$

$$R(0,\tau) = 1 - \sin\tau\left(\frac{1}{\tau} + \frac{\tau}{6}\right). \tag{21}$$

To remove the integration over p_3, we write the factor s^{-2} as $s^{-3/2}s^{-1/2}$ and note that $1/\sqrt{s}$ should be due the integration over p_3:

$$\frac{1}{\sqrt{s}} = \frac{e^{i\pi/4}}{\sqrt{\pi}}\int_0^\infty dp_3 e^{-isp_3^2}. \tag{22}$$

So

$$\ln C_{1n}^{*ren}(E=0,H) = -\frac{e^{i\pi/4}}{2\sqrt{\pi}}\int_0^\infty \frac{ds}{s^{3/2}}e^{-is(m^2+eH(2l+1)+p_3^2)}R(0,\tau). \tag{23}$$

(Substituting $s \to -it$ we see that expression (23) is purely imaginary.) From here, or from (21) we obtain in agreement with [8, 9]

$$i\sum_n \ln C_{1n}^{*ren}(E=0,H) = i\int\frac{dp_2}{2\pi}\int\frac{dp_3}{2\pi}\sum_{l=0}^\infty \ln C_{1n}^{*ren}(E=0,H) = \mathcal{L}_0 =$$

$$-\frac{eH}{16\pi^2}\int_0^\infty \frac{ds}{s^2\sin\tau}e^{-ism^2}R(0,\tau), \quad (L=T=1) \tag{24}$$

Relation (39) was used here and the sum over l was performed with the help of formula

$$\sum_{l=0}^{\infty} e^{-iseH(2l+1)} = \frac{1}{2i\sin eHs}. \tag{25}$$

3 Electron in the constant electromagnetic field

The Bogoliubov transformation has the form

$$\begin{gathered} {}_+\psi_n = c_{1n}\,{}^+\psi_n + c_{2n}\,{}^-\psi_n, \\ {}_-\psi_n = -c^*_{2n}\,{}^+\psi_n + c^*_{1n}\,{}^-\psi_n; \\ |c_{1n}|^2 + |c_{2n}|^2 = 1. \end{gathered} \tag{26}$$

For the constant electric field we have

$$c^*_{1n} = -i\sqrt{\frac{2\pi}{\varkappa}}\frac{e^{-\frac{\pi\varkappa}{2}}}{\Gamma(i\varkappa)}, \quad c_{2n} = e^{-\pi\varkappa}, \quad n = (p_1, p_2, p_3, r). \tag{27}$$

These Bogoliubov coefficients are independent of the spin state index $r = 1, 2$.

As in the scalar case, we start with [2]

$$< 0_{n\,out}|0_{n\,in} >= c^*_{1n}, \tag{28}$$

and

$$< 0_{out}|0_{in} >= \prod_n c^*_{1n} = e^{iW_{1/2}},\ W_{1/2} = \sum_n W_{1/2;n},\ W_{1/2;n} = -i\ln c^*_{1n} \tag{29}$$

From (27) we have

$$\ln c^*_{1n} = -\frac{i\pi}{2} + \frac{1}{2}\ln\frac{2\pi}{\varkappa} - \frac{\pi\varkappa}{2} - \ln\Gamma(i\varkappa). \tag{30}$$

The asymptotic expansion for $\Gamma(i\varkappa)$ is, see eq.(8.344) in [14] or eq. (6.1.40.) in [15]

$$\ln\Gamma(i\varkappa) = (i\varkappa - \frac{1}{2})\ln(i\varkappa) - i\varkappa + \frac{1}{2}\ln 2\pi + i\sum_{k=1}(-1)^k\frac{B_{2k}}{2k(2k-1)}(\varkappa)^{1-2k}. \tag{31}$$

From (30) and (31) we obtain

$$\ln C^*_{1n} \equiv \ln c^*_{1n} + i(\varkappa\ln\varkappa - \varkappa + \frac{\pi}{4}) = -i\sum_{k=1}(-1)^k\frac{B_{2k}}{2k(2k-1)}(\varkappa)^{1-2k}, \tag{32}$$

$$\ln C^{*ren}_{1n} \equiv \ln C^*_{1n} - \frac{i}{12\varkappa} = -i\sum_{k=2}(-1)^k\frac{B_{2k}}{2k(2k-1)}(\varkappa)^{1-2k}. \tag{33}$$

As in the scalar case we find

$$\ln C^*_{1n} = -\frac{1}{2}\int_0^\infty \frac{dx}{x}e^{-i2\varkappa x}(\coth x - \frac{1}{x}), \tag{34}$$

$$\ln C^{*ren}_{1n} = -\frac{1}{2}\int_0^\infty \frac{dx}{x} e^{-i2\varkappa x}\coth x[1-\tanh x(\frac{1}{x}+\frac{x}{3})], \tag{35}$$

Eq. (2.4.22.6) in [16] was used to verify (34), cf. the text before eq.(15).

The generalization of (35) for the presence of a constant magnetic field is straightforward. We rewrite it in the form ($x=\theta=eEs$)

$$\ln C^{*ren}_{1n}(E,H) = -\frac{1}{2}\int_0^\infty \frac{d\theta}{\theta} e^{-is(m^2+eH2l)}\coth\theta R(\theta,\tau),$$

$$n=(p_1,p_2,p_3,r);\ l=l_{min},l_{min}+1,\cdots,\quad l_{min}=0\quad \text{for } r=1,\ l_{min}=1\quad \text{for } r=2. \tag{36}$$

$R(\theta,\tau)$ may be taken from the Lagrange function [8, 9] ($\tau=eHs$)

$$R(\theta,\tau) = 1-\tan\tau\tanh\theta(\frac{1}{\theta\tau}+\frac{E^2-H^2}{3EH}). \tag{37}$$

Integrating over p_3 with the help the second equation in (12) we find

$$\int\frac{dp_3}{2\pi}\ln C^{*ren}_{1n} = -\frac{eE}{4\pi}\int_0^\infty \frac{d\theta}{\theta} e^{-is(m^2+eH2l)}\coth\theta R(\theta,\tau). \tag{38}$$

The subsequent integration over p_2 is performed according to formula similar to (12) [2]

$$\int dp_2 = eHL. \tag{39}$$

To sum up over r and l in (36), we use the formula obtainable from (25)

$$\sum_{r=1}^{2}\sum_{l_{min}}^{\infty} e^{-is2eHl} = -i\cot eHs. \tag{40}$$

So in agreement with the Lagrange function for the constant electromagnetic field [8, 9] we have

$$\sum_n \ln C^{*ren}_{1n} = i\frac{e^2EH}{8\pi^2}\int_0^\infty \frac{d\theta}{\theta} e^{-ism^2}\coth\theta\cot\tau R(\theta,\tau),\quad (L=T=1). \tag{41}$$

Now returning to (38), we can switch off the electric field

$$\int\frac{dp_3}{2\pi}\ln C^{*ren}_{1n} = -\frac{1}{4\pi}\int_0^\infty \frac{ds}{s^2} e^{-is(m^2+eH2l)}R(0,\tau),$$

$$R(0,\tau) = 1-\tan\tau(\frac{1}{\tau}-\frac{\tau}{3}); \tag{42}$$

l are given in (36). Using (22) we obtain as in the scalar case

$$\ln C^{*ren}_{1n}(E=0,H) = -\frac{e^{i\pi/4}}{2\sqrt{\pi}}\int_0^\infty \frac{ds}{s^{3/2}} e^{-is(m^2+p_3^2+eH2l)}R(0,\tau),$$

$$n=(p_1,p_2,p_3,r);\quad l=0,1,2,\cdots \text{ for } r=1;\quad l=1,2,\cdots \text{ for } r=2. \tag{43}$$

In the next Sections we give the heuristic derivation of $\ln C^{*ren}_{1n}$ not resorting to c^*_{1n}, but using the proper-time method The main problem here is due to the necessity to make renormalizations. We know how to renormalize $\mathcal{L}$ as a whole, but we have to renormalize a contribution to it from a particular state n. To do this we assume as before that the regularizer does not depend on n.

4 Scalar particle

We take the vector-potential of a constant electromagnetic field in the form

$$A_\mu = \delta_{\mu 2} H x_1 - \delta_{\mu 3} E t, \tag{44}$$

but start with the particle in a constant magnetic field, $E = 0$ in (44). The propagator with coinciding x and x' has the form (see for example [11])

$$G_0(x, x|E=0, H) = i\sqrt{\frac{eH}{\pi}} \Sigma_{l=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \times$$

$$\int_0^\infty ds \frac{D_l^2(\zeta)}{l!} \exp\{-is[m^2 + eH(2l+1) + p_3^2 - p_0^2]\}, \quad \zeta = \sqrt{2eH}(x_1 + \frac{p_2}{eH}). \tag{45}$$

According to (1) we have to integrate $\mathcal{L}_0$ and hence $G_0(x, x)$ over d^4x. The integration over x_1 is done with the help of formula

$$\int\limits_{-\infty}^{\infty} d\zeta D_l^2(\zeta) = \sqrt{2\pi} l!, \quad \text{or} \int\limits_{-\infty}^{\infty} dx_1 D_l^2(\zeta) = \left(\frac{\pi}{eH}\right)^{1/2} l!. \tag{46}$$

Integrating over p^0 and x_1, we obtain

$$\int_{-\infty}^{\infty} dx_1 G_0(x, x) = \frac{\exp[i\frac{3\pi}{4}]}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \Sigma_{l=0}^{\infty} \int_0^\infty \frac{ds}{\sqrt{s}} e^{-is[m^2 + eH(2l+1) + p_3^2]}. \tag{47}$$

As noted in [3] (see eq. (2.12) there), it follows from Schwinger results [8] that for scalar particle (boson)

$$-i\frac{\partial W_b}{\partial m^2} = \int d^4x G_b(x, x), \quad \text{or} \quad W_b = -i\int_{m^2}^{\infty} d\tilde{m}^2 \int d^4x G_b(x, x|\tilde{m}^2). \tag{48}$$

This means that $\mathcal{L}_0$ can be obtained from (47) by inserting $-1/s$ in the integrand. Inserting also the regularizer from (21), we get

$$iW_0(E=0, H) = i\mathcal{L}_0(E=0, H) = \frac{\exp[i\frac{\pi}{4}]}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \Sigma_{l=0}^{\infty} \int_0^\infty \frac{ds}{s^{3/2}} \times$$

$$e^{-is[m^2 + eH(2l+1) + p_3^2]} R(0, \tau) = -\int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int_{-\infty}^{\infty} \frac{dp_3}{2\pi} \Sigma_{l=0}^{\infty} \ln C_{1n}^{*ren}. \quad (L = T = 1) \tag{49}$$

Now for the constant electromagnetic field, described by the vector-potential (44), we insert in (2a) the expressions for the wave functions (see [2] with the modifications for $e' = -e = -|e|$) and use relation (93) in [11] (or relation similar to (96) below). Than we find

$$G_0(x, x|E, H) = \frac{e^{i3\pi/4}}{2\sqrt{\pi eE}} \int\limits_{-\infty}^{\infty} \frac{dp_2}{2\pi} \int\limits_{-\infty}^{\infty} \frac{dp_3}{2\pi} \Sigma_{l=0}^{\infty} \left(\frac{eH}{\pi}\right)^{1/2} \frac{D_l^2}{l!} \sqrt{2} \int\limits_0^{\infty} \frac{d\theta}{\sqrt{\sinh 2\theta}} \times$$

$$\exp[-i2\varkappa\theta - i\frac{T^2}{2\coth\theta}], \quad \theta = eEs, \quad T = \sqrt{2eE}(t - \frac{p_3}{eE}). \tag{50}$$

Integrating over x_1 (see (46)) and t, we get

$$\int\limits_{-\infty}^{\infty} dx_1 \int\limits_{-\infty}^{\infty} dt G_0(x, x|E, H) =$$

$$\frac{i}{2}\int_{-\infty}^{\infty}\frac{dp_2}{2\pi}\int_{-\infty}^{\infty}\frac{dp_3}{2\pi}\Sigma_{l=0}^{\infty}\int\limits_{0}^{\infty}\frac{ds}{\sinh(eEs)}\exp[-is(m^2 + eH(2l+1))]. \tag{51}$$

Going over from $G_0(x, x)$ to $\mathcal{L}_0$ is realized by inserting the factor $(-1/s)$ in the integrand in (51). Inserting also the regularizer $R(\tau, \theta)$, see eq. (37), we obtain

$$W_0(E, H) = i\int\limits_{-\infty}^{\infty}\frac{dp_2}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dp_3}{2\pi}\Sigma_{l=0}^{\infty}\ln C_{1n}^{*ren}$$

$$= -\frac{i}{2}\int\limits_{-\infty}^{\infty}\frac{dp_2}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dp_3}{2\pi}\Sigma_{l=0}^{\infty}\int\limits_{0}^{\infty}\frac{ds}{s\sinh\theta}\exp\{-is[m^2 + eH(2l+1)]\}R(\tau, \theta). \tag{52}$$

5 Spinor particle

First we consider the electron in the constant magnetic field, $E = 0$ in (44). The squared Dirac equation can be brought to the form

$$\{\frac{d^2}{d\zeta^2} - \frac{\zeta^2}{4} + \frac{p_0^2 - p_3^2}{2eH} - \frac{1}{2}\Sigma_3\}Z = 0, \quad \Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{53}$$

Here ζ is the same as in (45). We see that Z can be written as follows

$$Z = \mathrm{diag}(f_1, f_2, f_1, f_2)e^{i(p_2x_2 + p_3x_3 - p^0t)} \tag{54}$$

and f_1, f_2 have to satisfy the equation

$$\{\frac{d^2}{d\zeta^2} - \frac{\zeta^2}{4} + \frac{p_0^2 - p_3^2}{2eH} \mp \frac{1}{2}\}f_{1,2} = 0. \tag{55}$$

We choose $f_1 = D_{l-1}(\zeta)$, $f_2 = D_l(\zeta)$ in order that $p_\perp^2 = 2eHl$ in both cases. The solutions of the Dirac equation are obtained as the columns of the matrix [2]

$$(m - i\hat{\Pi})Z = 0, \quad \hat{\Pi} = i\gamma_\mu\Pi_\mu, \quad \Pi_\mu = -i\partial_\mu + eA_\mu. \tag{56}$$

Using the $\gamma-$matrices in the standard representation [4] we have

$$(m - i\hat{\Pi}) = \begin{pmatrix} m + \Pi^0 & 0 & -\Pi_3 & -\Pi_1 + i\Pi_2 \\ 0 & m + \Pi^0 & -\Pi_1 - i\Pi_2 & \Pi_3 \\ \Pi_3 & \Pi_1 - i\Pi_2 & m - \Pi^0 & 0 \\ \Pi_1 + i\Pi_2 & -\Pi_3 & 0 & m - \Pi^0 \end{pmatrix}. \tag{57}$$

In terms of ζ we get

$$\Pi_1 + i\Pi_2 = -i\sqrt{2eH}\left(\frac{d}{d\zeta} - \frac{\zeta}{2}\right), \quad \Pi_1 - i\Pi_2 = -i\sqrt{2eH}\left(\frac{d}{d\zeta} + \frac{\zeta}{2}\right). \tag{58}$$

Using also the relations

$$\left(\frac{d}{d\zeta} + \frac{\zeta}{2}\right) D_l(\zeta) = lD_{l-1}(\zeta), \left(\frac{d}{d\zeta} - \frac{\zeta}{2}\right) D_l(\zeta) = -D_{l+1}(\zeta), \tag{59}$$

we find

$$(m - i\hat{\Pi})Z =$$

$$\begin{pmatrix} (m+p^0)D_{l-1}(\zeta) & 0 & -p_3 D_{l-1}(\zeta) & il\sqrt{2eH}D_{l-1}(\zeta) \\ 0 & (m+p^0)D_l(\zeta) & -i\sqrt{2eH}D_l(\zeta) & p_3 D_l(\zeta) \\ p_3 D_{l-1}(\zeta) & -il\sqrt{2eH}D_{l-1}(\zeta) & (m-p^0)D_{l-1}(\zeta) & 0 \\ i\sqrt{2eH}D_l(\zeta) & -p_3 D_l(\zeta) & 0 & (m-p^0)D_l(\zeta) \end{pmatrix}. \tag{60}$$

Choosing the second and the first columns as ψ_1 and ψ_2 (subscripts 1 and 2 indicate spin states) and normalizing them, we get

$$_+\psi_1 = N_n \begin{bmatrix} 0 \\ (m+p^0)D_l(\zeta) \\ -il\sqrt{2eH}D_{l-1}(\zeta) \\ -p_3 D_l(\zeta) \end{bmatrix} e^{iq\cdot x}, \quad N_n = \left(\frac{eH}{\pi}\right)^{1/4} \sqrt{\frac{1}{2p^0(p^0+m)l!}},$$

$$p^0 = \sqrt{m^2 + 2eHl + p_3^2}, \quad q\cdot x = p_2 x_2 + p_3 x_3 - p^0 t,$$

$$n = (p_2, p_3, l, r), \quad \zeta = \sqrt{2eH}(x_1 + \frac{p_2}{eH}), \tag{61}$$

$$_+\psi_2 = N_n\sqrt{l} \begin{bmatrix} (m+p^0)D_{l-1}(\zeta) \\ 0 \\ p_3 D_{l-1}(\zeta) \\ i\sqrt{2eH}D_{l-1}(\zeta) \end{bmatrix} e^{iq\cdot x}, \quad l = 0, 1, 2, \cdots \tag{62}$$

As seen from (62) in this state l begins actually from unity. The negative-frequency solutions $_-\psi_n$ are obtained from (61-62) by substitution $q \to -q$. We note here that eqs. (61-62) differ from eq. (10.5.9) in [4] because there the authors assumed the charge of a spinor particle to be positive.

Having obtained the wave functions, we are going to find the contribution to $\mathcal{L}_{1/2}$ from each state ψ_n. For the field which does not create pairs, the propagator has the usual form

$$G_{1/2}(x, x') = i\Sigma_n \left\{ \begin{array}{ll} {}_+\psi_n(x)_+\bar{\psi}_n(x'), & t > t', \\ -{}_-\psi_n(x)_-\bar{\psi}_n(x'), & t < t, \end{array} \right. \quad \bar{\psi}_n = \psi_n^* \beta. \tag{63}$$

In the standard representation

$$\beta = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix}, \quad \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{64}$$

From (61) and (64) we find

$$\mathrm{tr}_+\psi_1(x)_+\bar{\psi}_1(x) = N_n^2\{[(m+p^0)^2 - p_3^2]D_l^2(\zeta) - 2eHl^2D_{l-1}^2(\zeta)\}. \tag{65}$$

Integrating over x_1, we obtain, see (46)

$$\int\limits_{-\infty}^{\infty} \mathrm{tr}_+\psi_1(x)_+\bar{\psi}_1(x) = \frac{m}{p^0}, \quad p^0 = \sqrt{m^2 + 2eHl + p_3^2}, \quad l = 0, 1, \cdots \tag{66}$$

Similarly, from (62) we get

$$\int\limits_{-\infty}^{\infty} \mathrm{tr}_+\psi_2(x)_+\bar{\psi}_2(x) = \frac{m}{p^0}, \quad l = 1, 2, \cdots \tag{67}$$

For the negative-frequency states we have to substitute $p^0 \to -p^0$. Now we can write

$$\frac{1}{|p^0|} = \frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi}} \int\limits_0^{\infty} \frac{ds}{\sqrt{s}} e^{-is(m^2+2eHl+p_3^2)}, \tag{68}$$

accommodating both lines in (63). Thus, from (63) and (66-68) we have

$$\int\limits_{-\infty}^{\infty} dx_1 G_{1/2}(x,x) = \Sigma_n \frac{e^{\frac{3i\pi}{4}}}{\sqrt{\pi}} m \int\limits_0^{\infty} \frac{ds}{\sqrt{s}} e^{-is(m^2+2eHl+p_3^2)}, \tag{69}$$

where $l = 0, 1, \cdots$ for $r = 1$ and $l = 1, 2, \cdots$ for $r = 2$. Next we use the analog of (48) for the electron

$$W_{1/2} = i\int_m^{\infty} d\tilde{m}\,\mathrm{Tr}G_{1/2}(x,x|\tilde{m}). \tag{70}$$

Here Tr means the integration over d^4x and the sum over spin states; as above we put $VT = 1$. Since

$$i\int\limits_m^{\infty} d\tilde{m}\tilde{m}e^{-is\tilde{m}^2} = \frac{e^{-ism^2}}{2s}, \tag{70a}$$

we see that $W_{1/2}$ may be obtained from (69) by inserting the factor $(\frac{1}{2ms})$ into the integrand. So we find

$$\mathcal{L}_{1/2} = \Sigma_n \frac{e^{\frac{3i\pi}{4}}}{2\sqrt{\pi}} \int\limits_0^{\infty} \frac{ds}{s^{3/2}} e^{-i(m^2+2eHl+p_3^2)} R(0,\tau). \tag{71}$$

This is in agreement with (43) and (29). To check this result, we integrate over $\frac{dp_2}{2\pi}$ with the help of (39), over $\frac{dp_3}{2\pi}$ with the help of (22) and use (40). Then, as expected, we get

$$\mathcal{L}_{1/2}(E=0,H) = \frac{eH}{8\pi^2} \int\limits_0^{\infty} \frac{ds}{s^2} e^{-ism^2} \cot\tau R(0,\tau), \tag{72}$$

see eq. (47) in Ch. 1 in the last Ref. in [9] for $E=0$.

Going over to the constant electromagnetic field described by vector-potential (44), we use $\gamma-$matrices in the spinor representation because then both α_3 and Σ_3 are diagonal. The squared Dirac equation has the form

$$(\Pi^2+m^2+g)Z=0, \quad g=\begin{pmatrix} (H-iE)\sigma_3 & 0 \\ 0 & (H+iE)\sigma_3 \end{pmatrix}, \tag{73}$$

Π_μ is defined in (56). So

$$Z=\mathrm{diag}(f_1,f_2,f_3,f_4)e^{i(p_2x_2+p_3x_3)}. \tag{74}$$

In terms of ζ and T (see (45) and (50)) we obtain the equation for f_1 and f_2

$$\{2eH[-\frac{\partial^2}{\partial\zeta^2}+\frac{\zeta^2}{4}\pm\frac{1}{2}]+2eE[\frac{\partial^2}{\partial T^2}+\frac{T^2}{4}\mp\frac{i}{2}]+m^2\}f_{1,2}=0. \tag{75}$$

Similarly, for f_3 and f_4

$$\{2eH[-\frac{\partial^2}{\partial\zeta^2}+\frac{\zeta^2}{4}\pm\frac{1}{2}]+2eE[\frac{\partial^2}{\partial T^2}+\frac{T^2}{4}\pm\frac{i}{2}]+m^2\}f_{3,4}=0. \tag{76}$$

From these equations we obtain

$$\begin{aligned}{}^+Z=\mathrm{diag}\{&D_{l-1}(\zeta)D_{-i\varkappa-1}(\chi),D_l(\zeta)D_{-i\varkappa}(\chi),D_{l-1}(\zeta)D_{-i\varkappa}(\chi),D_l(\zeta)D_{-i\varkappa-1}(\chi)\}\times\\ &e^{i(p_2x_2+p_3x_3)}, \quad \chi=e^{\frac{i\pi}{4}}T.\end{aligned} \tag{77}$$

Solutions of the Dirac equation with $\gamma-$matrices in spinor representation are obtained from equation

$$(m-i\hat{\Pi})Z=\begin{pmatrix} m & 0 & \Pi^0+\Pi_3 & \Pi_1-i\Pi_2 \\ 0 & m & \Pi_1+i\Pi_2 & \Pi^0-\Pi_3 \\ \Pi^0-\Pi_3 & -\Pi_1+i\Pi_2 & m & 0 \\ -\Pi_1-i\Pi_2 & \Pi^0+\Pi_3 & 0 & m \end{pmatrix}Z. \tag{78}$$

In terms of χ we have

$$\Pi^0\pm\Pi_3=-e^{-\frac{i\pi}{4}}\sqrt{2eE}\left(\frac{\partial}{\partial\chi}\pm\frac{\chi}{2}\right), \quad \chi=e^{\frac{i\pi}{4}}T. \tag{79}$$

Taking into account also (58-59) and the relations

$$\begin{aligned}(\Pi^0+\Pi_3)D_\nu(\chi)&=-e^{-\frac{i\pi}{4}}\nu\sqrt{2eE}D_{\nu-1}(\chi),\\ (\Pi^0-\Pi_3)D_\nu(\chi)&=e^{-\frac{i\pi}{4}}\sqrt{2eE}D_{\nu+1}(\chi),\end{aligned} \tag{80}$$

we find four columns of the matrix $(m-i\hat{\Pi})\,{}^+Z$

$$\begin{bmatrix} mD_{l-1}(\zeta)D_{-i\varkappa-1}(\chi) \\ 0 \\ e^{-\frac{i\pi}{4}}\sqrt{2eE}D_{l-1}(\zeta)D_{-i\varkappa}(\chi) \\ -i\sqrt{2eH}D_l(\zeta)D_{-i\varkappa-1}(\chi) \end{bmatrix}, \quad \begin{bmatrix} 0 \\ mD_l(\zeta)D_{-i\varkappa}(\chi) \\ il\sqrt{2eH}D_{l-1}(\zeta)D_{-i\varkappa}(\chi) \\ e^{\frac{i\pi}{4}}\varkappa\sqrt{2eE}D_l(\zeta)D_{-i\varkappa-1}(\chi) \end{bmatrix},$$

$$\begin{bmatrix} e^{\frac{i\pi}{4}}\varkappa\sqrt{2eE}D_{l-1}(\zeta)D_{-i\varkappa-1}(\chi) \\ i\sqrt{2eH}D_l(\zeta)D_{-i\varkappa}(\chi) \\ mD_{l-1}(\zeta)D_{-i\varkappa}(\chi) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -il\sqrt{2eH}D_{l-1}(\zeta)D_{-i\varkappa-1}(\chi) \\ e^{-\frac{i\pi}{4}}\sqrt{2eE}D_l(\zeta)D_{-i\varkappa}(\chi) \\ 0 \\ mD_l(\zeta)D_{-i\varkappa-1}(\chi) \end{bmatrix}.. \tag{81}$$

Here and below $e^{i(p_2x_2+p_3x_3)}$ is dropped for brevity. We denote $^+\psi_1$ ($^+\psi_2$) the fourth (first) column multiplied by the normalization factor

$$^+N_n = \exp(-\frac{\pi\varkappa}{4})(l!2eE)^{-1/2}(eH/\pi)^{\frac{1}{4}} \quad (^+N_n\sqrt{l}). \tag{82}$$

Next we consider the positive-frequency solution of (73) for $t \to -\infty$

$$_+Z = \mathrm{diag}\{D_{l-1}(\zeta)D_{i\varkappa}(\tau), D_l(\zeta)D_{i\varkappa-1}(\tau), D_{l-1}(\zeta)D_{i\varkappa-1}(\tau), D_l(\zeta)D_{i\varkappa}(\tau)\}. \tag{83}$$

Here $\tau = -e^{-i\frac{\pi}{4}}T$. In terms of this variable we have

$$\Pi^0 \pm \Pi_3 = -e^{\frac{i\pi}{4}}\sqrt{2eE}\left(\frac{\partial}{\partial\tau} \mp \frac{\tau}{2}\right). \tag{84}$$

Similarly to (80) we find

$$\begin{aligned} (\Pi^0 + \Pi_3)D_\nu(\tau) &= e^{\frac{i\pi}{4}}\sqrt{2eE}D_{\nu+1}(\tau), \\ (\Pi^0 - \Pi_3)D_\nu(\tau) &= -e^{\frac{i\pi}{4}}\nu\sqrt{2eE}D_{\nu-1}(\tau). \end{aligned} \tag{85}$$

With the help of these relations we get from (78) and (83) four columns of the matrix $(m - i\hat{\Pi})_+Z$

$$\begin{bmatrix} mD_{l-1}(\zeta)D_{i\varkappa}(\tau) \\ 0 \\ e^{-\frac{i\pi}{4}}\varkappa\sqrt{2eE}D_{l-1}(\zeta)D_{i\varkappa-1}(\tau) \\ -i\sqrt{2eH}D_l(\zeta)D_{i\varkappa}(\tau) \end{bmatrix}, \quad \begin{bmatrix} 0 \\ mD_l(\zeta)D_{i\varkappa-1}(\tau) \\ il\sqrt{2eH}D_{l-1}(\zeta)D_{i\varkappa-1}(\tau) \\ e^{\frac{i\pi}{4}}\sqrt{2eE}D_l(\zeta)D_{i\varkappa}(\tau) \end{bmatrix},$$

$$\begin{bmatrix} e^{\frac{i\pi}{4}}\sqrt{2eE}D_{l-1}(\zeta)D_{i\varkappa}(\tau) \\ i\sqrt{2eH}D_l(\zeta)D_{i\varkappa-1}(\tau) \\ mD_{l-1}(\zeta)D_{i\varkappa-1}(\tau) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -il\sqrt{2eH}D_{l-1}(\zeta)D_{i\varkappa}(\tau) \\ e^{-\frac{i\pi}{4}}\varkappa\sqrt{2eE}D_l(\zeta)D_{i\varkappa-1}(\tau) \\ 0 \\ mD_l(\zeta)D_{i\varkappa}(\tau) \end{bmatrix}. \tag{86}$$

Using again the fourth and the first columns, we have

$$_+\psi_1(x) = {}_+N_n \begin{bmatrix} -il\sqrt{2eH}D_{l-1}(\zeta)D_{i\varkappa}(\tau) \\ e^{-\frac{i\pi}{4}}\varkappa\sqrt{2eE}D_l(\zeta)D_{i\varkappa-1}(\tau) \\ 0 \\ mD_l(\zeta)D_{i\varkappa}(\tau) \end{bmatrix} e^{i(p_2x_2+p_3x_3)}, \tag{87}$$

$$_+\psi_2(x) = {}_+N_n\sqrt{l} \begin{bmatrix} mD_{l-1}(\zeta)D_{i\varkappa}(\tau) \\ 0 \\ e^{-\frac{i\pi}{4}}\varkappa\sqrt{2eE}D_{l-1}(\zeta)D_{i\varkappa-1}(\tau) \\ -i\sqrt{2eH}D_l(\zeta)D_{i\varkappa}(\tau) \end{bmatrix} e^{i(p_2x_2+p_3x_3)}. \tag{88}$$

Here ${}_+N_n = {}^+N_n/\sqrt{\varkappa}$, see (82).

Now we note that ${}_-Z$ (${}^-Z$) can be obtained from ${}^+Z$ (${}_+Z$) by the substitution $\chi \to -\chi$ ($\tau \to -\tau$). To obtain ${}_-\psi-$functions from the corresponding ${}^+\psi$-functions, we have besides these substitution also change the sign of $\sqrt{2eE}$ in the columns; this is because of relations (see (79) and (80))

$$(\Pi^0 + \Pi_3)D_\nu(\pm\chi) = \mp e^{-\frac{i\pi}{4}}\nu\sqrt{2eE}D_{\nu-1}(\pm\chi),$$

$$(\Pi^0 - \Pi_3)D_\nu(\pm\chi) = \pm e^{-\frac{i\pi}{4}}\sqrt{2eE}D_{\nu+1}(\pm\chi). \tag{89}$$

Thus,

$${}_-\psi_1(x) = {}_-N_n \begin{bmatrix} -il\sqrt{2eH}D_{l-1}(\zeta)D_{-i\varkappa-1}(-\chi) \\ -e^{-\frac{i\pi}{4}}\sqrt{2eE}D_l(\zeta)D_{-i\varkappa}(-\chi) \\ 0 \\ mD_l(\zeta)D_{-i\varkappa-1}(-\chi) \end{bmatrix} e^{i(p_2x_2+p_3x_3)}, \tag{90}$$

$${}_-\psi_2(x) = {}_-N_n\sqrt{l} \begin{bmatrix} mD_{l-1}(\zeta)D_{-i\varkappa-1}(-\chi) \\ 0 \\ -e^{-\frac{i\pi}{4}}\sqrt{2eE}D_{l-1}(\zeta)D_{-i\varkappa}(-\chi) \\ -i\sqrt{2eH}D_l(\zeta)D_{-i\varkappa-1}(-\chi) \end{bmatrix} e^{i(p_2x_2+p_3x_3)}, \quad {}_-N_n = {}^+N_n, \tag{91}$$

and similarly for ${}^-\psi_1$ and ${}^-\psi_2$.

We note by the way that the wave functions for an electron in a constant electric field were written down in [2] using $\gamma-$matrices in the standard representation. Acting on these functions by an operator

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{I} & \mathrm{I} \\ \mathrm{I} & -\mathrm{I} \end{pmatrix}, \quad \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we get the solutions in spinor representation. Taking into account the magnetic field is realized by the substitutions

$$e^{ip_2x_2}\{1, p_1 - ip_2, p_1 + ip_2\} \to \left(\frac{eH}{\pi}\right)^{1/4}\frac{1}{\sqrt{l!}}\{D_l(\zeta), -il\sqrt{2eH}D_{l-1}(\zeta), i\sqrt{2eH}D_{l+1}(\zeta)\}$$

for $r = 1$. For $r = 2$ we have to replace in these substitutions l by $l - 1$.

The electron propagator has the form

$$G_{1/2}(x, x') = i\Sigma_n c_{1n}^{*-1}\begin{cases} {}^+\psi_n(x){}_+\bar{\psi}_n(x'), & t > t', \\ -{}_-\psi_n(x){}^-\bar{\psi}_n(x'), & t < t' \end{cases}. \tag{92}$$

Here $\bar{\psi} = \psi^*\beta$ and for the constant electromagnetic field $n = (p_2, p_3, l, r)$, c_{1n}^* is given in (27). In the spinor representation

$$\beta = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix}, \quad \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{93}$$

so that

$$(a_1, a_2, a_3, a_4)\beta = (a_3, a_4, a_1, a_2).$$

Now using (81-82) and (87) we obtain

$$\mathrm{tr}({}^{+}\psi_1(x)\ {}_{+}\bar{\psi}_1(x)) = \left(\frac{eH}{\pi}\right)^{1/2} \frac{m \exp[-\frac{\pi\varkappa}{2}]}{l!\sqrt{2eE\varkappa}} \times$$

$$D_l^2(\zeta)\{e^{-i\frac{\pi}{4}} D_{-i\varkappa}(\chi) D_{-i\varkappa}(-\chi) + e^{i\frac{\pi}{4}} \varkappa D_{-i\varkappa-1}(\chi) D_{-i\varkappa-1}(-\chi)\}. \tag{94}$$

Integrating over x_1, we get, see (46)

$$\int\limits_{-\infty}^{\infty} dx_1 \mathrm{tr}({}^{+}\psi_1(x)\ {}_{+}\bar{\psi}_1(x)) = \frac{m \exp[-\frac{\pi\varkappa}{2}]}{\sqrt{2eE\varkappa}} \{e^{-i\frac{\pi}{4}} D_{-i\varkappa}(\chi) D_{-i\varkappa}(-\chi) +$$

$$e^{i\frac{\pi}{4}} \varkappa D_{-i\varkappa-1}(\chi) D_{-i\varkappa-1}(-\chi)\}, \quad \varkappa = \frac{m^2 + 2eHl}{2eE}, \quad l = 0, 1, \cdots. \tag{95}$$

For $r = 2$ we obtain the same expression, but with $l = 1, 2, \cdots$

Next we multiply (95) by i/c_{1n}^* according to (92) and make use of the relation, see eq. (93) in [11] with $-i\varkappa \to -i\varkappa + 1/2$

$$\Gamma(i\varkappa) D_{-i\varkappa}(\chi) D_{-i\varkappa}(-\chi) = \sqrt{2} \int\limits_0^{\infty} \frac{d\theta}{\sqrt{\sinh 2\theta}} e^{-i2\varkappa\theta + \theta - \frac{i}{2}T^2 \tanh\theta}, \quad \theta = eEs, \tag{96}$$

and relation obtainable from this by the substitution $i\varkappa \to i\varkappa + 1$

$$\Gamma(i\varkappa + 1) D_{-i\varkappa-1}(\chi) D_{-i\varkappa-1}(-\chi) = \sqrt{2} \int\limits_0^{\infty} \frac{d\theta}{\sqrt{\sinh 2\theta}} e^{-i2\varkappa\theta - \theta - \frac{i}{2}T^2 \tanh\theta}. \tag{97}$$

Now we get from (95-97)

$$\int\limits_{-\infty}^{\infty} dx_1 \frac{i}{c_{1n}^*} \mathrm{tr}({}^{+}\psi_1(x)\ {}_{+}\bar{\psi}_1(x)) = -\frac{m \exp(-i\frac{\pi}{4})}{\sqrt{2\pi eE}} \times$$

$$\int\limits_0^{\infty} \frac{d\theta}{\sqrt{\sinh 2\theta}} 2\cosh\theta e^{-i2\varkappa\theta - \frac{i}{2}T^2 \tanh\theta}, \quad T = \sqrt{2eE}\left(t - \frac{p_3}{eE}\right). \tag{98}$$

Integrating this expression over t, we obtain

$$\int\limits_{-\infty}^{\infty} dt \int\limits_{-\infty}^{\infty} dx_1 \frac{i}{c_{1n}^*} \mathrm{tr}({}^{+}\psi_1(x)\ {}_{+}\bar{\psi}_1(x)) =$$

$$im \int\limits_0^{\infty} ds \coth eEs e^{-is(m^2 + 2eHl)}, \quad l = 0, 1, 2, \cdots \tag{99}$$

For $r = 2$ we have the same expression, but with $l = 1, 2, \cdots$.

Taking into account the remarks after eq. (70a) and inserting the regularizer $R(\theta,\tau)$ in the integrand, we obtain the contribution to $\mathcal{L}_{1/2}$ from the state $n=(p_2,p_3,l,r)$. Summing up over l and r (see (40)) and integrating over $\frac{dp_2}{2\pi}$ and $\frac{dp_3}{2\pi}$ (see (39) and (12)), we obtain in agreement with (41)

$$\mathcal{L}_{1/2}=\frac{e^2HE}{8\pi^2}\int\limits_0^\infty\frac{ds}{s}e^{-ism^2}\coth\theta\cot\tau R(\theta,\tau). \tag{100}$$

Finally we note that for $H=0$ we have instead of (99)

$$\int\limits_{-\infty}^{\infty}dt\frac{i}{c_{1n}^*}\mathrm{tr}({}_+\psi_1(x)\ {}^+\bar{\psi}_1(x))=$$

$$im\int\limits_0^\infty ds\coth eEse^{-is(m^2+p_1^2+p_2^2)},\quad l=0,1,2,\cdots \tag{101}$$

Inserting $1/(2ms)$ and $R(\theta,\tau)$, we get the agreement with (35).

6 Vector boson

The propagator and the effective Lagrange function for the vector boson with gyromagnetic ratio $g=2$ in a constant electromagnetic field was obtained by Vanyashin and Terentyev [17]. In another form the propagator was found by the author [16]. In this latter paper there is a misprint in eq. (73), where the argument of sin and cos should be 2τ not τ. Besides, the statement, that the divergence term in the expression for the current in eq. (38) does not contribute, is not true when the magnetic field is present; this, however, is of no consequence since the expression was used only for the normalization of wave functions.

From the results of Vanyashin and Terentyev it follows that the relation (48) holds also for the vector boson, if we mean that $G_b=G^\mu{}_\mu$. Using (48) we can reproduce the expression for $\mathcal{L}_1$ in [17] starting from our propagator. Indeed, our result for

$$G^\mu{}_\mu=\frac{e^2EH}{16\pi^2}\int_C\frac{ds}{\sinh\theta\sin\tau}e^{-ism^2}\times$$

$$\{2\cos2\tau+2\cosh2\theta-\frac{i}{m^2}[eH\cot\tau+eE\coth\theta]\} \tag{102}$$

can be written in a more simple form, if we note that

$$\frac{d}{ds}\frac{1}{\sinh\theta\sin\tau}=-\frac{1}{\sinh\theta\sin\tau}[eH\cot\tau+eE\coth\theta]\},\quad\tau=eHs,\quad\theta=eEs. \tag{103}$$

Then we can integrate by parts the term in the square brackets in (102)

$$-\frac{ie^2EH}{16\pi^2m^2}\int_C\frac{ds}{\sinh\theta\sin\tau}e^{-ism^2}[eH\cot\tau+eE\coth\theta]\}\Longrightarrow$$

$$-\frac{e^2EH}{16\pi^2}\int_C \frac{ds}{\sinh\theta\sin\tau}e^{-ism^2} \tag{104}$$

Here we discarded divergent term independent of E and H. So (102) is equivalent to

$$\frac{e^2EH}{16\pi^2}\int_C \frac{ds}{\sinh\theta\sin\tau}e^{-ism^2}\{2\cos 2\tau+2\cosh 2\theta-1\}. \tag{105}$$

Inserting $(-1/s)$ in the integrand, we get eq. (21) in [17] and we agree with the subsequent formulas in that paper.

Returning to our present problem we note that for the constant electric field c^*_{1n} is independent of the polarization state of vector boson and is the same as in the scalar case [11]. Nevertheless $\mathrm{Im}\mathcal{L}_1$ is not simply $3\mathrm{Im}\mathcal{L}_0$ [17]. Thus, the knowledge of c^*_{1n} is of no use in obtaining $\ln C^{*ren}_{1n}$. Resorting to the proper-time method, we find that the problem is more difficult then in the previous cases. As seen already from (102), the dependence on m^2 is more complicated here and the contributions from the electric and magnetic fields are not factorized in the proper-time integrand. For these reasons we consider here only the constant magnetic field.

From [11] we have for the spin states $r=1,2,3$

$$_+\psi^\mu_1(x)\ {}_+\psi^*_{1\mu}(x)=$$

$$\left(\frac{eH}{\pi}\right)^{1/2}\frac{1}{2|p^0|l!}\frac{1}{(l+1)(m^2+eHl)}\{-(l+1)^2eHD^2_l(\zeta)+[m^2+eH(2l+1)]D^2_{l+1}(\zeta)\}; \tag{106}$$

$$_+\psi^\mu_2(x)\ {}_+\psi^*_{2\mu}(x)=\left(\frac{eH}{\pi}\right)^{1/2}\frac{1}{2|p^0|l!}D^2_l(\zeta); \tag{107}$$

$$_+\psi^\mu_3(x)\ {}_+\psi^*_{3\mu}(x)=$$

$$\left(\frac{eH}{\pi}\right)^{1/2}\frac{1}{2|p^0|l!}\frac{l}{2m^2(m^2+eHl)}\{-2eH[m^2+eH(2l+1)]D^2_l(\zeta)++[eHD_{l+1}(\zeta)$$
$$-(m^2+eHl)D_{l-1}(\zeta)]^2+[eHD_{l+1}(\zeta)+(m^2+eHl)D_{l-1}(\zeta)]^2]\}. \tag{108}$$

Integrating with the help of (46) the expressions in (106-108) over x_1, we get in all three cases $1/(2|p^0|)$, but

$$l=l_{min},l_{min}+1,\cdots,\quad l_{min}=\begin{cases}-1, & r=1\\ 0, & r=2\\ 1, & r=3.\end{cases} \tag{109}$$

The vector boson propagator has the form [11]

$$G^{\mu\nu}_1(x,x')=i\int\limits_{-\infty}^{\infty}\frac{dp_2}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dp_3}{2\pi}\Sigma^3_{r=1}\Sigma^\infty_{l_{min}}\begin{cases} {}_+\psi_n(x)^\mu{}_+\psi^{*\nu}_n(x'), & t>t',\\ {}_-\psi^(_nx)^\mu{}_-\psi^{*\nu}_n(x'), & t<t'\end{cases} \tag{110}$$

We see from (68), (48) and the above results that the contribution to $\mathcal{L}_1$ from the state with quantum numbers $n = (p_2, p_3, l, r)$ is

$$i \ln c^*_{1n} = -i \frac{e^{\frac{i\pi}{4}}}{2\sqrt{\pi}} \int_0^\infty \frac{ds}{(s)^{3/2}} e^{-is[m^2+eH(2l+1)+p_3^2]}. \tag{111}$$

The sum over r and l is performed with the help of formula obtainable from (25)

$$\Sigma_{r=1}^3 \Sigma_{l_{min}}^\infty e^{-iseH(2l+1)} = \frac{1}{2i \sin eHs}[1 + 2\cos 2eHs]. \tag{112}$$

To integrate over $\frac{dp_2}{2\pi}$ and $\frac{dp_3}{2\pi}$, we use (39) and (12). Then, inserting $R(\tau)$, we obtain

$$\Sigma_n W_{spin1,n} = -\frac{eH}{16\pi^2} \int_0^\infty \frac{ds}{s^2 \sin\tau} e^{-ism^2}[3 - 4\sin^2\tau]R(\tau). \tag{113}$$

$R(\tau)$ is defined in accordance with [17]

$$\frac{3 - 4\sin^2\tau}{\sin\tau} \to 3\left(\frac{1}{\sin\tau} - \frac{1}{\tau} - \frac{\tau}{6}\right) - 4(\sin\tau - \tau) = \frac{3 - 4\sin^2\tau}{\sin\tau} R(\tau). \tag{114}$$

From here

$$R(\tau) = 1 - \frac{\sin\tau}{3 - \sin^2\tau}\left(\frac{3}{\tau} - \frac{7}{2}\tau\right), \quad R(\tau)|_{\tau \ll 1} = \frac{29}{120}\tau^4. \tag{115}$$

Hence from (111) we have

$$i \ln C^{*ren}_{1n} = -i \frac{e^{\frac{i\pi}{4}}}{2\sqrt{\pi}} \int_0^\infty \frac{ds}{(s)^{3/2}} e^{-is[m^2+eH(2l+1)+p_3^2]} R(\tau). \tag{116}$$

l are given in (109). Substituting $\tau \to -it$ and rotating the integration contour, we see that $\ln C^{*ren}_{1n}$ is real as it should be for the magnetic field.

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